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Application of the field method to the non-linear theory of vibrations

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Abstract

This paper deals with the generalization of the field method to weakly non-linear vibrational systems with one degree of freedom. The field co-ordinate and field momentum approaches are combined with the method of multiple time scales in order to obtain the amplitude and phase of oscillations in the first approximation. Apart from the fact that the algorithm of obtaining motion is condensed for the cases of a free vibrational system and system with slowly varying parameters, in this paper the field method is extended to a parametrically excited system and system with external excitation.

1. Introduction

The well-known Hamilton–Jacobi theory is suitable for studying conservative systems, which are completely describable by Hamiltonians of systems. However, for the case of non-conservative ones it fails to be applicable. That fact was motivating for Vujanovic and his co-workers to formulate a parallel method, the field method [1–6], so as not to have such a limiting condition.

Although these two methods have some similarities, there are also some differences between them. First, in Hamilton's mechanics generalized momenta are considered to be the field of the gradient vector of the scalar function possessing the physical meaning of the Hamiltonian action. In Vujanovic' method the role of the field has one of the state variables, i.e., the generalized co-ordinate (that is known as the generalized co-ordinate approach) or the generalized momentum (generalized momentum approach). Therefore, the field is a quantity directly involved in the dynamics of a system. Secondly, neither of these two methods looks for a solution directly from the equations of motion. They find it through the complete integral of a certain partial differential equation of the first order. The Hamilton–Jacobi equation is strictly non-linear with respect to the momentum gradient and contains $(n+1)$ independent variables (n—the number of degrees of

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freedom of the system), while, the so-called basic field equation in Vujanovic' theory is quasi-linear with $2n$ independent variables. Although a general method for finding its integral does not exist, quasi-linearity frequently makes it more easily solvable. The same problem arises in the Hamilton–Jacobi method, especially when the separation of variables is not possible. Furthermore, the comparison of these two methods shows that motion in both cases follows from an algebraic procedure on the complete integral.

Finally, what attracts attention is the applicability of these two methods to the theory of vibrations. Among methods based on the Hamilton–Jacobi theory some approximate methods are developed. The method of variation of the constants [7], the von Zeipel method [7,8] and the method of Hori and Deprit [9] are the most popular. The first one (thanks to the combination with the method of averaging) gives the solution in the first approximation only. The method developed by von Zeipel includes the canonical transformations of the old variables to the new ones. The lack of this method is that some steps of the procedure contain both old and new variables simultaneously. Hori and Deprit developed the method based on Lee's transformations, which is canonically invariant but very strenuous.

While the Hamilton–Jacobi method has not been related to the method of multiple scales, Vujanovic's field method gave approximate asymptotic solution of rheo-linear and weakly non-linear problems by applying two time scales expansion to the basic field concept [3–5]. The procedure given in these references may seem a bit far-fetched, especially because of the variety of formal transformations during the process of finding the solution.

This paper is the generalization of the Vujanovic's method for the non-Hamiltonian system whose mathematical model represents a one-degree-of-freedom mechanical system:

$$\dot{x} = p,$$
$$\dot{p} = -\omega^2 x + \varepsilon F(x, p, t), \tag{1}$$

where x is a generalized co-ordinate, p is a generalized momentum, ω is a parameter of the system, t is the time, ε is a small parameter ($0 < \varepsilon \ll 1$), F is a given non-linear function depending on x, p and t, while an overdot denotes differentiation with respect to time.

This generalization leads to the first order differential equation for the amplitude and phase of motion in the first approximation. These equations have forms equivalent to ones usually obtained by some other approximate methods, such as the method of multiple scales [7] or Bogoliubov–Mitropolski method [9]. This fact sets off the field method expanding its primary purpose to a wider perspective.

Apart from the fact that the algorithm of obtaining motion is condensed for the cases of a free vibrational system and system with slowly varying parameters, in this paper the Vujanovic's method is extended to a parametrically excited system and system with external excitation, which have not been studied by the field method before.

2. Basis of the field method

Let the differential equations of a mechanical system be of the following form:

$$\dot{x}_1 = X_1(t, x_1, x_2), \quad \dot{x}_2 = X_2(t, x_1, x_2), \tag{2}$$

where x_1, x_2 are the state variables of the system (generalized co-ordinate or momentum).

The basis supposition of the field method is that one of the state variables can be interpreted as a field depending on time t and the rest of variables, i.e.

$$x_1 = U(t, x_2). \tag{3}$$

Instead of finding the solution of Eq. (2) directly, Vujanovic [1–5] suggests deriving it through the basic equation:

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x_2} X_2 - X_1(t, U, x_2) = 0, \tag{4}$$

obtained by differentiating Eq. (3) with respect to time and using the Eqs. (2).

Its complete solution

$$x_1 = U(t, x_2, C_1, C_2), \tag{5}$$

depends on two arbitrary constants C_i.

If the system is formulated as an initial value problems:

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \tag{6}$$

one of the constants can be expressed as

$$C_1 = C_1(C_2, x_{10}, x_{20}). \tag{7}$$

Then solution (5) transforms into the conditioned form solution:

$$x_1 = \bar{U}(t, x_2, C_2, x_{10}, x_{20}). \tag{8}$$

Theorem 1 (Vujanovic's theorem [1–5]). *The solution of system* (1) *with initial conditions* (6) *can be found from the conditioned form solution* (8) *and the algebraic equation*:

$$\frac{\partial \bar{U}}{\partial C_2} = 0. \tag{9}$$

The previously given concept of the field method is applied to the weakly non-linear vibrational system (1). Non-linearity of the system requires that this concept be confined with the method of multiple scales. Depending on which state variable has been chosen for the field, the field co-ordinate and field momentum approaches are applied. Since both of approaches have their own features, they will be demonstrated separately.

3. Generalization of a field co-ordinate approach

First, system (1) is analyzed for the case when, in accordance with the notation from the previous section, $x_1 \equiv x, x_2 \equiv p$. Choosing the co-ordinate x for the field

$$x = U(t, p), \tag{10}$$

it depends on time t and the momentum p. The basic equation (4) is of the form

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial p}[-\omega^2 U + \varepsilon F(U, p, t)] - p = 0. \tag{11}$$

To find the complete solution (5) of this weakly non-linear equation in the closed form is frequently impossible. An approximate solution can be accomplished by applying the technique of multiple scales in the first approximation and introducing two independent variables [7]:

$$T = t, \quad \tau = \varepsilon t. \tag{12}$$

Thus, the field and the momentum can be developed asymptotically in powers of the small parameter ε as follows:

$$U(T, p, \varepsilon) = U_0(T, \tau, p_0) + \varepsilon U_1(T, \tau, p_1) + \cdots, \tag{13}$$

$$p(T, \varepsilon) = p_0(T, \tau) + \varepsilon U_1(T, \tau) + \cdots. \tag{14}$$

Further, Vujanovic [3,4] imposes the requirement that $\partial U/\partial p$ does not depend on the step of approximation, which means that the field component U_l $(l = 0, 1)$ are being changed with the components p_l linearly and uniquely:

$$\frac{\partial U}{\partial p} = \frac{\partial U_0}{\partial p_0} = \frac{\partial U_1}{\partial p_1} = \cdots. \tag{15}$$

Using Eqs. (12)–(15) the basic equation transforms, after equating the terms containing the same powers of the small parameter, into the following system:

$$\frac{\partial U_0}{\partial T} - \omega^2 \frac{\partial U_0}{\partial p_0} U_0 - p_0 = 0, \tag{16}$$

$$\frac{\partial U_1}{\partial T} - \omega^2 \frac{\partial U_1}{\partial p_1} U_1 - p_1 = -\frac{\partial \bar{U}_0}{\partial \tau} - \frac{\partial \bar{U}_0}{\partial p_0} F(\bar{U}_0|_{p_0}, p_0, T, \tau). \tag{17}$$

The left sides of these equations are similar to each other, and consequently their solutions have the same form. The right side of Eq. (17) depends on the quantities that should be found on the basis of the previously calculated solution of Eq. (16). It means that all of them should be expressed in the conditioned form and solved along trajectory, i.e. for the value of the first component of the momentum.

In accordance with Refs. [3–5] the solution of Eq. (16) is

$$\bar{U}_0 = \frac{p_0}{\omega} \tan(\omega T + C_2) + \frac{A(\tau)\cos C_2 - B(\tau)\sin C_2}{\cos(\omega T + C_2)}. \tag{18}$$

Applying the Vujanovic's theorem mathematically described by Eq. (9) it follows that

$$p_0 = \omega[-A(\tau)\sin \omega T + B(\tau)\cos \omega T]. \tag{19}$$

In addition, the solution along trajectory is

$$\bar{U}_0|_{p_0} = A(\tau)\cos \omega T + B(\tau)\sin \omega T. \tag{20}$$

The detailed analysis of the application of the suggested solution (18) given in Refs. [1,3–5] points out the existence of many formal transformations. It is better to write it down as

$$\bar{U}_0 = \frac{p_0}{\omega} \tan(\omega T + C_2) + \frac{a(\tau)\cos(\beta(\tau) - C_2)}{\cos(\omega T + C_2)}. \tag{21}$$

Applying condition (9), one obtains

$$p_0 = -\omega a(\tau)\sin(\omega T + \beta(\tau)), \tag{22}$$

and the solution along the trajectory is

$$\bar{U}_0|_{p_0} = a(\tau)\cos(\omega T + \beta(\tau)). \tag{23}$$

Obviously, by using the new form of the conditioned solution (21), the solution in the first approximation for a vibrational system is obtained in the usual form. The functions $a(t)$ and $\beta(t)$ are to be calculated and they have meaning of the amplitude and phase of vibrations. They will be found from the requirement of no secular terms.

In analogy to Eq. (21), the complete solution of the basic equation (17) is

$$U_1 = \frac{p_1}{\omega}\tan(\omega T + C_2) + \frac{D(T,\tau)}{\cos(\omega T + C_2)}, \tag{24}$$

where $D(T,t)$ is a new unknown function.

Obviously, the further consideration depends on the form of a non-linear function in Eq. (1). Hence, some forms will be assigned to it and solution (23) will be found completely.

3.1. Parametrically excited systems

Since the problem of parametric resonance arises in many branches of physics and engineering, an attempt to treat this system by the field method will be made.

Let system (1) be modelled by a modified Mathieus equation, which means that the non-linear function has the form

$$F(U,t,p) = -2\alpha_1 U\cos 2t + f(U,p), \tag{25}$$

where α_1 is a positive given constant and f is a non-linear function of U and p.

The consideration will be restricted to the case of principal resonance that is $\omega^2 \approx 1$.

So, after substituting Eqs. (21)–(24) into Eq. (17) one gets

$$\begin{aligned}\frac{\mathrm{d}D}{\mathrm{d}T} = &-\frac{\mathrm{d}a}{\mathrm{d}\tau}\cos(\beta - C_2) + a\frac{\mathrm{d}\beta}{\mathrm{d}\tau}\sin(\beta - C_2)\\ &+\frac{\alpha_1\cos 2T}{\omega}\sin 2(\omega T + C_2) + \frac{\sin(\omega T + C_2)}{\omega}f(\bar{U}_0|_{p_0}, p_0).\end{aligned} \tag{26}$$

To express the nearness of ω to 1, a detuning parameter is introduced:

$$1 = \omega + \varepsilon\sigma, \quad \sigma = O(1). \tag{27}$$

Putting it into Eq. (26) and separating the terms containing $\cos C$ and $\sin C$, respectively, leads to

$$\begin{aligned}0 = &-\frac{\mathrm{d}a}{\mathrm{d}\tau}\cos\beta + a\frac{\mathrm{d}\beta}{\mathrm{d}\tau}\sin\beta - \frac{a\alpha_1}{2\omega}\sin(2\sigma\tau - \beta)\\ &-\frac{\sin\omega T}{\omega}f(a(\tau)\cos(\omega T + \beta(\tau)), -\omega a(\tau)\sin(\omega T + \beta(\tau))),\end{aligned} \tag{28}$$

$$0 = -\frac{\mathrm{d}a}{\mathrm{d}\tau}\sin\beta - a\frac{\mathrm{d}\beta}{\mathrm{d}\tau}\cos\beta + \frac{a\alpha_1}{2\omega}\cos(2\sigma\tau - \beta) - \frac{\cos\omega T}{\omega} f(a(\tau)\cos(\omega T + \beta(\tau)), -\omega a(\tau)\sin(\omega T + \beta(\tau))). \tag{29}$$

At this point, among some algebraic transformations, the method of averaging to the function f over a period 2π will be applied.

Consequently, one has

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -\frac{a\alpha_1}{2\omega}\sin(2\sigma\tau - 2\beta) - \frac{1}{2\pi\omega}\int_0^{2\pi} \sin\phi f(a(\tau)\cos\phi, -\omega a(\tau)\sin\phi)\,\mathrm{d}\phi, \tag{30}$$

$$a\frac{\mathrm{d}\beta}{\mathrm{d}\tau} = \frac{a\alpha_1}{2\omega}\cos(2\sigma\tau - 2\beta) - \frac{1}{2\pi\omega}\int_0^{2\pi} \cos\phi f(a(\tau)\cos\phi, -\omega a(\tau)\sin\phi)\,\mathrm{d}\phi, \tag{31}$$

where $\phi = \omega T + \beta$.

This pair of equations is of the same form as that found by the method of multiple scales [7] for the case $\alpha_1 = 1$.

The previously presented procedure shows advantage and elegance of the field method. The fact that the field method is so effective and quick puts it shoulder to shoulder to the others widely used approximate asymptotic methods, such as Krilov–Bogoliubov and Linstead–Poincare method.

3.1.1. Example 1. Parametrically excited non-linear system

To analyze the influence of a parametric excitation to system (25) by the field method, the function $f(x, p)$ is considered as

$$f(U, p) = -2\delta p + \sum_{r=2}^{5} \alpha_r U^r, \tag{32}$$

where δ, α_1 and α_r are constant parameters. Note that α_r can be positive (soft spring) or negative (hard spring).

Substituting Eq. (32) into Eqs. (30) and (31), the following system of the first order differential equations is obtained:

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -\frac{a\alpha_1}{2\omega}\sin(2\sigma\tau - 2\beta) - \delta a, \tag{33}$$

$$a\frac{\mathrm{d}\beta}{\mathrm{d}\tau} = \frac{a\alpha_1}{2\omega}\cos(2\sigma\tau - 2\beta) - \frac{1}{\omega}\left[\frac{3}{8}\alpha_3 a^3 + \frac{5}{16}\alpha_5 a^5\right]. \tag{34}$$

It is seen that the amplitude of vibrations is not directly affected by non-linear rigidity, but it is indirectly through the phase of vibrations, depending only on the cubic term and the term of the fifth power.

System (33), (34) is suitable for analyzing steady state solution $\mathrm{d}a/\mathrm{d}\tau = 0$, $\mathrm{d}\psi/\mathrm{d}\tau = 0$, appearing when

$$\sin\psi = -\frac{2\omega\delta}{\alpha_1}, \tag{35}$$

$$\cos\psi = \frac{2\sigma\omega}{\alpha_1} + \frac{2}{\alpha_1}\left[\frac{3}{8}\alpha_3 a^3 + \frac{5}{16}\alpha_5 a^5\right], \tag{36}$$

where $\psi = 2\sigma\tau - 2\beta$.

By squaring and adding them, the frequency—response equation in the first approximation is obtained:

$$1 = \frac{4\delta^2}{\alpha_1^2} + \frac{4}{\alpha_1^2}\left[\sigma + \left[\frac{3}{8}\alpha_3 a^2 + \frac{5}{16}\alpha_5 a^4\right]\right]^2. \tag{37}$$

In Ref. [7] such equation is analyzed for the case when $\alpha_5 = 0$. Now examine the influence of this parameter on the response of the system. The corresponding frequency–response curve is plotted in Fig. 1. It is seen that for the case of a hard spring, the steady state amplitude increases if the amplitude of excitation decreases and a detuning parameter increases. However, for the case of a soft spring the opposite is true. Namely, decrease of a detuning parameter and increase of the amplitude of excitation make the amplitude of vibrations higher.

To prove the exactness of the procedure, free non-linear damped vibrations as a special case will be considered.

Special case: *free non-linear damped vibrations*. If $\alpha_1 = 0$ Eqs. (33) and (34) give the first order differential equations defining the amplitude and phase of free non-linear damped vibrations:

$$\frac{da}{d\tau} = -\delta a, \tag{38}$$

$$a\frac{d\beta}{d\tau} = -\frac{1}{\omega}\left[\frac{3}{8}\alpha_3 a^3 + \frac{5}{16}\alpha_5 a^5\right], \tag{39}$$

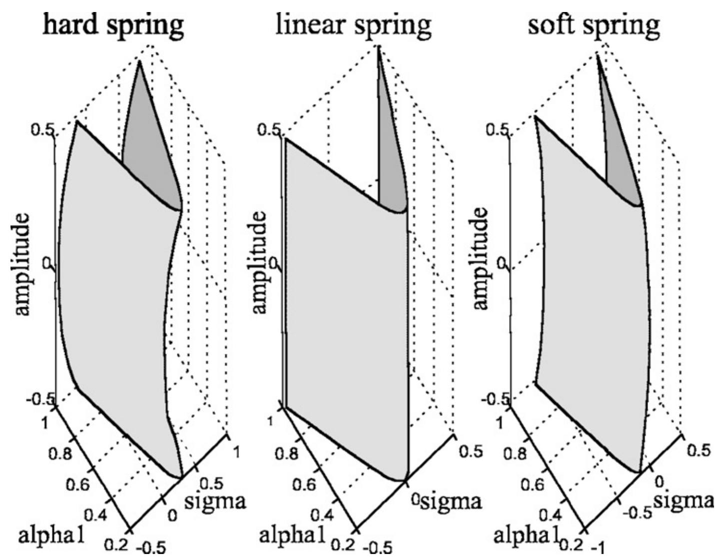

Fig. 1. Amplitude of excitation (alpha 1)-frequency (sigma)-response (amplitude) curves for parametrically excited non-linear system.

whose solution is

$$\bar{U}_0|_{p_0} = a\cos(\omega T + \beta(\tau)),$$
$$a = a_0 e^{-\varepsilon\delta t}, \quad \beta = \frac{3\alpha_3}{16\omega\delta} a_0^2(e^{-2\varepsilon\delta t} - 1) + \frac{5\alpha_5}{64\omega\delta} a_0^4(e^{-4\varepsilon\delta t} - 1) + \beta_0, \tag{40}$$

In Eqs. (40), a_0 is an initial amplitude, while $\beta(0) = \beta_0$.

The analytically obtained amplitude–time function is plotted in Fig. 2 for different values of the small parameter. It is the envelope for curves obtained numerically.

3.2. Forced oscillations

Consider system (1) under the influence of an external excitation, taking the function F in the form

$$F = -k\cos\Omega t + f(U, p), \tag{41}$$

where k and Ω are constants, while f is a non-linear function of the field U and momentum p.

It is also assumed that primary resonance appears, i.e.

$$\Omega = \omega + \varepsilon\sigma, \quad \sigma = O(1). \tag{42}$$

Substituting Eqs. (21)–(24) and (42) into Eq. (17) one has

$$\frac{dD}{dT} = -\frac{da}{d\tau}\cos(\beta - C_2) + a\frac{d\beta}{d\tau}\sin(\beta - C_2) - \frac{k}{\omega}\cos(\omega T + \sigma\tau)\cos(\omega T + C_2)$$
$$- \frac{\sin(\omega T + C_2)}{\omega} f(\bar{U}_0|_{p_0}, p_0). \tag{43}$$

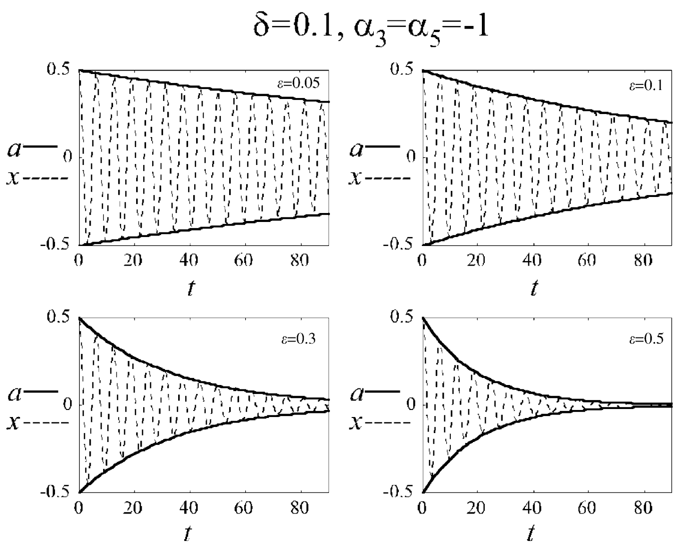

Fig. 2. Comparison between numerical solution $x(t)$ and analytical solution $a(t)$ for different values of the small parameter for free non-linear damped vibrations.

Elimination of the secular terms by equating all terms next to cos C and sin C to zero, gives

$$0 = -\frac{da}{d\tau}\cos\beta + a\frac{d\beta}{d\tau}\sin\beta + \frac{k}{2\omega}\sin\sigma\tau - \frac{\sin\omega T}{\omega}f(\bar{U}_0|_{p_0}, p_0), \tag{44}$$

$$0 = -\frac{da}{d\tau}\sin\beta - a\frac{d\beta}{d\tau}\cos\beta - \frac{k}{2\omega}\cos\sigma\tau - \frac{\cos\omega T}{\omega}f(\bar{U}_0|_{p_0}, p_0). \tag{45}$$

Averaging the right sides of Eqs. (44) and (45) leads to the first order differential equations for the amplitude and phase of vibrations:

$$\frac{da}{d\tau} = \frac{k}{2\omega}\sin(2\sigma\tau - \beta) - \frac{1}{2\pi\omega}\int_0^{2\pi}\sin\phi f(a(\tau)\cos\phi, -\omega a(\tau)\sin\phi)\,d\phi, \tag{46}$$

$$a\frac{d\beta}{d\tau} = -\frac{k}{2\omega}\cos(2\sigma\tau - \beta) - \frac{1}{2\pi\omega}\int_0^{2\pi}\cos\phi f(a(\tau)\cos\phi, -\omega a(\tau)\sin\phi)\,d\phi, \tag{47}$$

where $\phi = \omega T + \beta$.

Obviously, they have the general form of the equations discussed in Ref. [5] by the methods of Krilov–Bogoliubov–Mitropolski. They prove that the field method can be used for obtaining these types of equations, which are, among the rest, suitable for studying steady state motion.

3.2.1. *Example 2. Primary resonance in non-linear system*

Consider the forced response of a single-degree-of-freedom system (1), (41) for the case when

$$f(U, p) = -2\delta p + \sum_2^5 \alpha_r U^r, \tag{48}$$

where δ and α_r are constant parameters. The constants α_r can be positive for the soft spring or negative for the hard spring.

There are many studies dealing with this type of system [7–9]. It will be shown that the field method can also be applied to the study of this system.

Substitution of it into Eqs. (46) and (47) gives

$$\frac{da}{d\tau} = \frac{k}{2\omega}\sin(2\sigma\tau - \beta) - \delta a, \tag{49}$$

$$a\frac{d\beta}{d\tau} = -\frac{k}{2\omega}\cos(2\sigma\tau - \beta) - \frac{1}{\omega}\left[\frac{3}{8}\alpha_3 a^3 + \frac{5}{16}\alpha_5 a^5\right]. \tag{50}$$

System (49), (50) can be further used for studying steady state solution $da/d\tau = 0$, $d\gamma/d\tau = 0$, which is described as

$$2\omega\delta a = k\sin\gamma, \tag{51}$$

$$2\omega a\sigma + \left[\tfrac{3}{4}\alpha_3 a^3 + \tfrac{5}{8}\alpha_5 a^5\right] = -k\cos\gamma, \tag{52}$$

where $\gamma = \sigma\tau - \beta$.

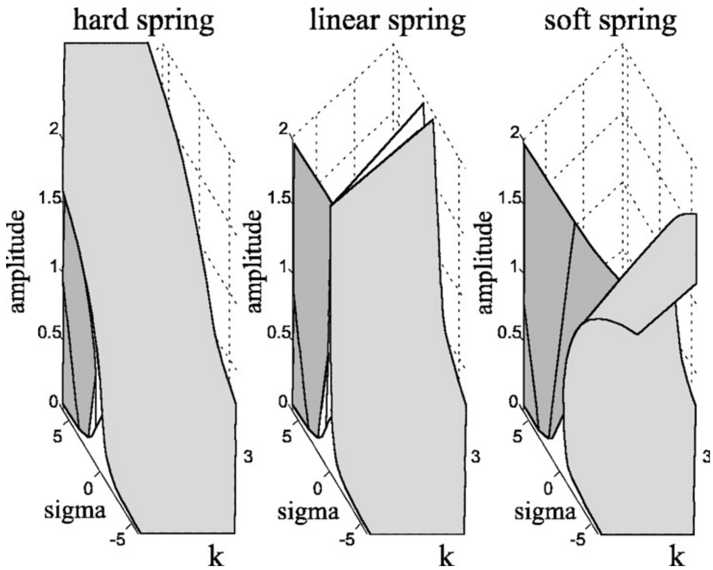

Fig. 3. Frequency (sigma)-amplitude of excitation (k)-response (amplitude) curves for forced non-linear vibrations.

Thus, the frequency–response equation of this system is

$$\frac{k^2}{4\omega^2} = \left(\delta^2 + \left[\sigma + \frac{\frac{3}{4}\alpha_3 a^2 + \frac{5}{8}\alpha_5 a^4}{2\omega}\right]^2\right)a^2. \tag{53}$$

Fig. 3 shows how the amplitude varies with the amplitude and phase of excitation. For all cases (hard spring $\alpha_3 < 0$, $\alpha_5 < 0$, linear spring $\alpha_3 = \alpha_5 = 0$ and soft spring $\alpha_3 > 0$, $\alpha_5 > 0$) cusps have appeared. Hence, the jump phenomenon, caused by non-linear phase–amplitude interaction in Eq. (53) is realized. The non-linearity bends the curve surface to the right for hard and to the left for soft spring.

Special case: *free non-linear damped vibrations*. If $k = 0$ Eqs. (49) and (50) define the amplitude and phase of free non-linear damped vibrations and have form (38), (39), derived for non-linear free oscillations. It proves the correctness of the method discussed in this paper as well as its generality.

4. Generalization of a field momentum approach

As the second case, the interpretation of the momentum p as a field (3) depending on time t and the co-ordinate x:

$$p = \Phi(t, x), \tag{54}$$

is considered in order to obtain the first order differential equations for the amplitude and phase of vibrations for the weakly non-linear system (1).

The corresponding basic equation (4), obtained by the total time derivation of Eq. (54) and using (1), is

$$\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x}\Phi + \omega^2 x - \varepsilon F(x, \Phi, t) = 0. \tag{55}$$

In analogy with the previously presented procedure and the definitions for fast and slow time given by Eq. (12), the approximate solutions for the field Φ and x in terms of the different scales are

$$\Phi(t, x, \varepsilon) = \Phi_0(T, \tau, x_0) + \varepsilon \Phi_1(T, \tau, x_1) + \cdots, \tag{56}$$

$$x(t, \varepsilon) = x_0(T, \tau) + \varepsilon x_1(T, \tau) + \cdots. \tag{57}$$

The next assumption, taken in analogy with Eq. (15), is that component Φ_l $(l = 0, 1)$ are being changed with respect to x_l linearly and uniquely:

$$\frac{\partial \Phi}{\partial x} = \frac{\partial \Phi_0}{\partial x_0} = \frac{\partial \Phi_1}{\partial x_1} = \cdots, \tag{58}$$

This supposition is very important for simplifying the further procedure.

In addition, validity of Eq. (54) orders the following compatibility conditions:

$$\Phi_0(x_0 T, \tau) = \frac{\partial x_0}{\partial T}, \tag{59}$$

$$\Phi_1(x_1 T, \tau) = \Phi_1^*(x_1, T, \tau) + \frac{\partial x_0}{\partial \tau}, \tag{60}$$

where $\Phi_1^* = \partial x_1 / \partial T$.

They are obtained by finding the total time derivative of Eq. (57) and equating it with Eq. (56) for all powers of ε.

With Eqs. (56)–(60), the basic equation (55) gives the following system of the partial differential equations:

$$\frac{\partial \Phi_0}{\partial T} + \frac{\partial \Phi_0}{\partial x_0}\Phi_0 + \omega^2 x_0 = 0, \tag{61}$$

$$\frac{\partial \Phi_1^*}{\partial T} + \frac{\partial \Phi_1^*}{\partial x_1}\Phi_1^* + \omega^2 x_1 = -\frac{\partial \bar{\Phi}_0}{\partial \tau} - \frac{\partial^2 x_0}{\partial T \partial \tau} - \frac{\partial x_0}{\partial \tau}\frac{\partial \bar{\Phi}_0}{\partial x_0} + F(x_0, \bar{\Phi}_0|_{p_0}, T, \tau). \tag{62}$$

The algorithm for obtaining an asymptotic solution in the first approximation is analogous to that described in the field co-ordinate approach. Namely, the first thing to do is to find the complete solution of Eq. (61) and then calculate the right side of Eq. (62) on the basis of the elimination of the secular terms.

The conditioned form solution, according to Refs. [3,4], is

$$\bar{\Phi}_0 = -x_0 \omega \tan(\omega T + C_2) + \frac{A(\tau)\sin C + B(\tau)\cos C_2}{\cos(\omega T + C_2)}. \tag{63}$$

It will be shown that assuming it in the form

$$\bar{\Phi}_0 = -x_0 \omega \tan(\omega T + C_2) + \frac{a(\tau)\omega \sin(\beta(\tau) - C_2)}{\cos(\omega T + C_2)}, \tag{64}$$

considerably simplifies the process of solution. In Eq. (64), $a(t)$ and $\beta(t)$ are the unknown functions.

The application of the Vujanovic's theorem described by Eqs. (9)–(64) gives

$$x_0 = a(\tau)\cos(\omega T + \beta(\tau)). \tag{65}$$

This solution has the well-known form of a solution in the first approximation.

The solution along trajectory is

$$\bar{\Phi}_0|_{x_0} = -a(\tau)\omega\sin(\omega T + \beta(\tau)). \tag{66}$$

Solutions (65) and (66) are completely in agreement with those obtained by the field co-ordinate approach (22), (23) and no complications are included despite the occurrence of the compatibility conditions.

Assume the solution of the left side of Eq. (62) to be as follows:

$$\Phi_1 = -x_1\omega\tan(\omega T + C_2) + \frac{D(T,\tau)}{\cos(\omega T + C_2)}, \tag{67}$$

where $D(T,t)$ is an unknown function. Further consideration will be continued after defining the form of a non-linear function in Eq. (1).

4.1. Free non-linear vibrations

Free weakly non-linear oscillations are studied in Refs. [4,5] for the case of a non-linear quadratic term in the mathematical model of the system. Although the obtained results are in agreement with those found by some other methods, it is noticeable that the algorithm includes a number of steps confusing the reader.

Consequently, a condensed procedure for obtaining the solution will be developed by applying the field momentum approach for system (1), where $F = F(x,p)$. Substituting this function and Eqs. (65)–(67) into Eq. (62), gives:

$$\begin{aligned}\frac{\mathrm{d}D}{\mathrm{d}T} = {}&\frac{\mathrm{d}a}{\mathrm{d}\tau}\sin(\beta - C_2) + a\frac{\mathrm{d}\beta}{\mathrm{d}\tau}\cos(\beta - C_2) + \frac{\mathrm{d}a}{\mathrm{d}\tau}\sin(2\omega T + \beta + C_2)\\ &+ a\frac{\mathrm{d}\beta}{\mathrm{d}\tau}\cos(2\omega T + \beta + C_2) - \frac{\cos(\omega T + C_2)}{\omega}F(x_0, \bar{\Phi}_0|_{p_0}).\end{aligned} \tag{68}$$

The requirement of removing the secular terms on the left side of Eq. (68) includes equating all terms containing $\cos C$ and $\sin C$ to zero. Thus,

$$\begin{aligned}0 = {}&\frac{\mathrm{d}a}{\mathrm{d}\tau}\sin\beta + a\frac{\omega}{2}\frac{\mathrm{d}\beta}{\mathrm{d}\tau}\cos\beta\\ &+ \frac{\cos\omega T}{\omega}F(a(\tau)\cos(\omega T + \beta(\tau)), -\omega a(\tau)\sin(\omega T + \beta(\tau))),\end{aligned} \tag{69}$$

$$\begin{aligned}0 = {}&-\frac{\mathrm{d}a}{\mathrm{d}\tau}\cos\beta + a\frac{\omega}{2}\frac{\mathrm{d}\beta}{\mathrm{d}\tau}\sin\beta\\ &- \frac{\sin\omega T}{\omega}F(a(\tau)\cos(\omega T + \beta(\tau)), -\omega a(\tau)\sin(\omega T + \beta(\tau))).\end{aligned} \tag{70}$$

After averaging, the first order differential equations for the amplitude and phase are obtained:

$$\frac{da}{d\tau} = -\frac{1}{2\pi\omega}\int_0^{2\pi} \sin\phi F(a(\tau)\cos\phi, -\omega a(\tau)\sin\phi)\, d\phi, \tag{71}$$

$$\frac{d\beta}{d\tau} = -\frac{1}{2\pi\omega}\int_0^{2\pi} \cos\phi F(a(\tau)\cos\phi, -\omega a(\tau)\sin\phi)\, d\phi, \tag{72}$$

where $\phi = \omega T + \beta$. Comparing these solutions with those obtained in Refs. [8,9], one finds that they are equivalent.

It is also noticeable that Eqs. (30) and (31) for the absence of a parametric excitation $\alpha_1 = 0$ and Eqs. (46) and (47) for the absence of an external excitation $k = 0$ are equivalent to Eqs. (71) and (72). This is another fact contributing to the generality of the field method.

Obviously, the application of the field momentum approach gives the required solution very quickly, on its own way. Its technique is made to be schematic, which can now be used in the study of non-linear vibration problems without any, previously existing prejudice.

4.2. Non-linear oscillator with slowly varying parameters

In Ref. [4], a linear oscillator with slowly varying frequency is studied by taking the field co-ordinate as a field. This consideration will be extended to a non-linear system by choosing the field momentum for the field and showing that the procedure presented in this paper keeps its formalism.

So, now study system (1) written in form (2) where $\omega = \omega(\tau)$ and $F = F(x, p, \tau)$, while the slow time is $\tau = \varepsilon t$ and $\varepsilon \ll 1$.

However, the fast time scale, as suggested in Ref. [8] should be determined as

$$T = \frac{1}{\varepsilon} g(\tau). \tag{73}$$

The asymptotic representations for Φ and x are assumed in form (57).

Using Eqs. (56)–(60) and (73), the basic equation (55) transforms into

$$\frac{\partial \Phi_0}{\partial T}\frac{dg}{d\tau} + \frac{\partial \Phi_0}{\partial x_0}\Phi_0 + \omega^2(\tau)x_0 = 0, \tag{74}$$

$$\frac{\partial \Phi_1^*}{\partial T}\frac{dg}{d\tau} + \frac{\partial \Phi_1^*}{\partial x_1}\Phi_1^* + \omega^2(\tau)x_1 = -\frac{\partial \bar{\Phi}_0}{\partial \tau} - \frac{\partial^2 x_0}{\partial T\partial\tau}\frac{dg}{d\tau} - \frac{\partial x_0}{\partial \tau}\frac{\partial \bar{\Phi}_0}{\partial x_0} + F(x_0, \bar{\Phi}_0|_{p_0}, T, \tau). \tag{75}$$

The supposed form of the complete solution of (74) is

$$\Phi_0 = -x_0\lambda(\tau)\tan(T + C_2) + \frac{M(\tau)}{\cos(T + C_2)}, \tag{76}$$

where $\lambda(\tau)$ and $M(\tau)$ are to be found.

After substituting Eq. (76) into Eq. (74), one has

$$0 = x_0\left[-\frac{dg}{d\tau}\lambda(\tau) + \omega^2(\tau) + \tan^2(T + C_2)\left(\lambda^2(\tau) - \lambda(\tau)\frac{dg}{d\tau}\right)\right] + M(\tau)\frac{\sin(T + C_2)}{\cos(T + C_2)}\left(\frac{dg}{d\tau} - \lambda(\tau)\right). \quad (77)$$

Since this relation should be satisfied for all x_0, $(T + C)$ and M, the asked functions are obtained:

$$\lambda(\tau) = \omega(\tau), \quad \frac{dg}{d\tau} = \omega(\tau), \quad (78)$$

allowing the process of finding the solution to be continued. Thus, the conditioned form solution is

$$\bar{\Phi}_0 = -x_0\omega(\tau)\tan(T + C_2) - \frac{a(\tau)\omega(\tau)\sin(\beta(\tau) - C_2)}{\cos(T + C_2)}, \quad (79)$$

while the complete solution for the first component of the field will be taken in the following form:

$$\Phi_1^* = -x_1\omega(\tau)\tan(T + C_2) + \frac{D(T,\tau)}{\cos(T + C_2)}. \quad (80)$$

Solution (79) differs from one given in Ref. [4], but it will be shown that the form is more suitable and, as it was already proven in the previous section, simplifies the consideration making it more familiar.

Application of Eq. (9) expressing Vujanovic's theorem to Eq. (79) leads to the solution in form Eq. (65), while the complete solution of Eq. (74) is Eq. (66). Using these relations, Eq. (75) transforms into

$$\begin{aligned}\omega(\tau)\frac{dD}{dT} = {} & \frac{d\omega}{d\tau}a\cos(T + \beta)\sin(C_2 + \beta) \\ & + \omega(\tau)\frac{da}{d\tau}\cos(T + \beta)\sin(\beta + C_2) \\ & + a\omega(\tau)\frac{d\beta}{d\tau}\cos(\beta - C_2) + a\omega(\tau)\frac{d\beta}{d\tau}\cos(2t + \beta + C_2) \\ & - \frac{\cos(\omega T + C_2)}{\omega}F(x_0, \bar{\Phi}_0|_{p_0}).\end{aligned} \quad (81)$$

Eliminating the secular terms, it follows that

$$0 = \frac{a}{2}\frac{d\omega}{d\tau} + \frac{da}{d\tau}\omega + \sin(T + \beta)F(a(\tau)\cos(T + \beta(\tau)), -\omega a(\tau)\sin(T + \beta(\tau))), \quad (82)$$

$$0 = a\omega\frac{d\beta}{d\tau} - \cos(T + \beta)F(a(\tau)\cos(T + \beta(\tau)), -\omega a(\tau)\sin(T + \beta(\tau))). \quad (83)$$

After some transformations, including averaging, one finds

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -\frac{a}{2\omega}\frac{\mathrm{d}\omega}{\mathrm{d}\tau} - \frac{1}{2\pi\omega}\int_0^{2\pi} \sin\phi f(a(\tau)\cos\phi, -\omega a(\tau)\sin\phi)\,\mathrm{d}\phi, \tag{84}$$

$$a\frac{\mathrm{d}\beta}{\mathrm{d}\tau} = -\frac{1}{2\pi\omega}\int_0^{2\pi} \cos\phi f(a(\tau)\cos\phi, -\omega a(\tau)\sin\phi)\,\mathrm{d}\phi, \tag{85}$$

where $\phi = T + \beta$.

The forms of Eqs. (84) and (85) are equivalent to the ones in Refs. [8,9] and for the case when ω is a constant they become equal to Eqs. (71) and (72) derived for the case of free vibrational system.

4.2.1. Example 3: a wagon-measuring mechanism system

As the example of a system with slowly varying parameters, a wagon-measuring mechanism system [10] will be studied. The mathematical model of this system for the non-linear model of rigidity, has form (1), where

$$\omega^2(\tau) = \frac{k_1}{m_0(1+\tau)}, \tag{86}$$

$$F(x,p,\tau) = \frac{p}{(1+\tau)} + \frac{c_1 p}{\varepsilon m_0(1+\tau)} - \frac{k_1}{q(1+\tau)}\left(x + \frac{qu}{k_1}\right)^3 - \frac{u}{(1+\tau)}, \tag{87}$$

while k_1, m_0, c_1, q and u are constant parameters of the system, $\varepsilon \ll 1$ $(\varepsilon = q/m_0)$ and τ is slow time.

Fast time, in accordance with Eqs. (73) and (74), is

$$T = \frac{2}{\varepsilon}\frac{k_1}{m_0}(\sqrt{1+\tau} - 1). \tag{88}$$

Thus, system (84) and (85) is

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -\frac{a}{4(1+\tau)} - \frac{1}{2\pi(1+\tau)}\int_0^{2\pi} \Delta_1 \sin\phi\,\mathrm{d}\phi, \tag{89}$$

$$a\frac{\mathrm{d}\beta}{\mathrm{d}\tau} = -\frac{1}{2\pi\omega}\int_0^{2\pi} \Delta_2 \cos\phi\,\mathrm{d}\phi, \tag{90}$$

where Δ_1 and Δ_2 stand for

$$\Delta_1 = \left(\frac{\omega a}{1+\tau}\sin\phi + \frac{c_1\omega a}{\varepsilon m_0(1+\tau)}\sin\phi + \frac{q^2u^3}{k_1^2(1+\tau)}\right),$$
$$\Delta_2 = \left(\frac{k_1}{1+\tau}\left[\frac{a^3\cos^3\phi}{q} + \frac{3qu^2 a\cos\phi}{k_1^2}\right]\right),$$

while $\phi = \beta + T$. Note that in system (89), (90) all terms become equal to zero after integration and hence are neglected.

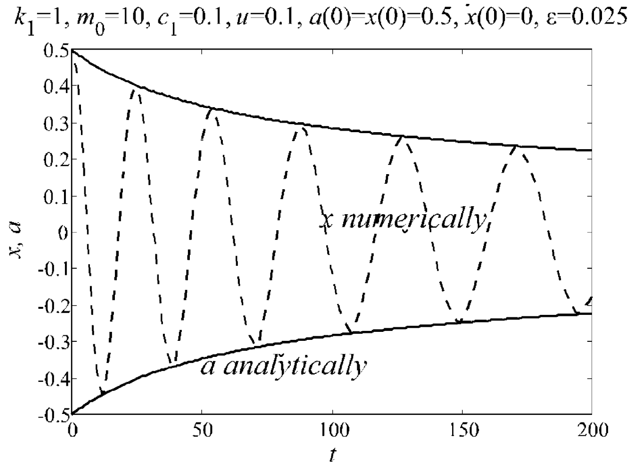

Fig. 4. Comparison between the numerical solution $x(t)$ and analytical one $a(t)$.

So, motion of system (1), (86), (87) with initial conditions $x(0) = a_0$, $\dot{x}(0) = 0$ is

$$x = a(\tau)\cos\left(\frac{2}{\varepsilon}\sqrt{\frac{k_1}{m_0}}(\sqrt{1+\tau} - \beta)\right), \tag{91}$$

$$a(\tau) = \frac{1}{(1+\tau)^{1/4+c_1/(2\varepsilon m_0)}}\left[\frac{q^2u^3\sqrt{m_0}}{k_1^{5/2}}\frac{1}{\frac{3}{4}+c_1/(2\varepsilon m_0)}(1-(1+\tau)^{1/4+c_1/(2\varepsilon m_0)}) + a_0\right].$$

It is seen that the non-linearity affects the amplitude directly. Its influence on amplitude appears as an additional term beside the term that contains the coefficient of internal damping c_1, increasing it.

Fig. 4 shows a good agreement between the numerical solution of system (1), (86), (87) and analytical solution (91) for the set of parameters written above the picture.

Special case: *linear system*. Analyze the case when the stiffness is modeled linearly, i.e.,

$$\dot{x} = p, \quad \dot{p} = -\frac{k_1}{m_0(1+\tau)}x - \varepsilon\frac{p}{(1+\tau)}. \tag{92}$$

According to the previously given field method formalism, motion of system (92) with initial conditions $x(0) = a_0$, $\dot{x}(0) = 0$ is,

$$x = a(\tau)\cos\left(\frac{2}{\varepsilon}\sqrt{\frac{k_1}{m_0}}(\sqrt{1+\tau} - 1)\right), \quad a(\tau) = \frac{a_0}{(1+\tau)^{1/4}}. \tag{93}$$

Fig. 5 shows the relative error between the numerical solution of Eq. (92) and the analytically obtained solution (93) for the set of parameters $k_1 = 1$, $m_0 = 10$, $x(0) = a_0 = 0.5$, $\dot{x}(0) = 0$ and different values of the small parameter ε. It is obvious that the agreement between these two solutions is acceptable and that the error gets smaller as time passes.

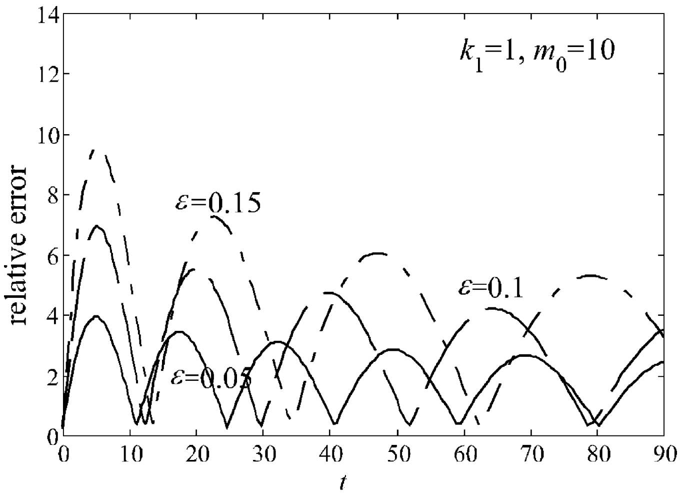

Fig. 5. Relative error [%] between the numerical and analytical solution for different values of the small parameter ε.

5. Conclusion

In this paper, generalization of the field method, through its approaches—field co-ordinate and field momentum, to non-Hamiltonian systems is carried out. The method is applied to different types of weakly non-linear vibration problems: forced, parametrically excited, free vibrational system and system with slowly varying parameters.

It is concluded:

- In spite of usual consideration being a complication if some method reduces solving the system of ordinary differential equations to solving a partial differential equation, it has been shown that in the case of the field method, such reduction, together with applying more suitable form of the complete solution of the basic partial equation, gives the sought solution elegantly and quickly.
- Combining the concept of the field method with multiple scales, the first order differential equations for the amplitude and phase of vibrations are obtained. They have the same form as the equations obtained by some others methods (Bogoliubov–Mitropolski, method of multiple scales).
- Elegance and schematics of the algorithm of the field method set off this method as a suitable for studying weakly non-linear systems.

Thus, the field method excels in its primary purpose to treat non-conservative systems in Hamiltonian mechanics and generalizes.

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